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# Time Optimal Control of Neutral Systems with Control Integral Constraint

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This note considers the time optimal problem for a linear neutral system with a control integral constraint. A maximum principle is derived.

## INTRODUCTION

Optimal control problems associated with certain functional differential equations have been studied in [1]–[3]. However, little attention seems to have been directed towards neutral systems. This work derives a maximum principle for the time optimal problem for a linear neutral system having an integral constraint on the control. The results presented can be considered an extension to those given by Neustadt [4] for a system represented by ordinary differential equations. It should be possible to weaken the assumptions by using the approach suggested in [5], but this has not been attempted.

Some aspects of optimal control of neutral systems have also been investigated in [6].

## PROBLEM STATEMENT

The system considered is represented by

$$\dot{x}(t) = A_1(t)\, x(t) + A_2(t)\, x(t-\tau) + A_3(t)\, \dot{x}(t-\tau) + B(t)\, u(t). \tag{1}$$

Given $x(t) = X(t)$, $-\tau \leqslant t \leqslant 0$, where $X(t)$ is continuously differentiable, it is desired to transfer the system to a given point $\eta \in R^n$ in minimum time.

## ASSUMPTIONS

(i) $A_1(t)$, $A_2(t)$, $A_3(t)$ are continuous $n \times n$ matrices, $A_3(t)$ is continuously differentiable, and $B(t)$ is a continuous $n \times m$ matrix.

(ii) The $m$-vector control $u(t)$ is measurable and satisfies the following constraints:

$$|u_i(t)| \leqslant 1, \qquad i = 1, 2, \ldots, m, \qquad \text{for all } t, \tag{2}$$

$$\int_0^{t^*} \phi(u(t))\, dt \leqslant L, \tag{3}$$

where $L$ is a specified positive number, and $\phi$ is described below. Denote the unit cube $|u_i| \leqslant 1$ by $U$.

(iii) The function $\phi(u)$ is defined, continuous, not constant, and non-negative for all $u$ which belong to an open set containing $U$. Also $\phi(u)$ is convex and takes its maximum value at each vertex of $U$.

## Formulation of the Problem

Define an augmented state variable $y(t)$ as follows:

$$y(t) = \begin{bmatrix} x(t) \\ y_{n+1}(t) \end{bmatrix},$$

where $y_{n+1}(t)$ satisfies the differential equation,

$$\dot{y}_{n+1}(t) = \phi\{u(t)\}, \qquad t \geqslant 0, \qquad y_{n+1}(0) = 0.$$

It can be shown that (1) has the solution,

$$x(t) = x_0(t) + \int_0^t Y(s, t)\, B(s)\, u(s)\, ds$$

where the $(n \times n)$ matrix $Y(s, t)$ is discontinuous (see Appendix) and satisfies

$$\begin{aligned} \frac{\partial}{\partial s} \{Y(s, t) - Y(s + \tau, t)\, A_3(s + \tau)\} \\ = -Y(s, t)\, A_1(s) - Y(s + \tau, t)\, A_2(s + \tau), \qquad 0 \leqslant s \leqslant t - \tau, \end{aligned} \tag{4}$$

$$\partial Y(s, t)/\partial s = -Y(s, t)\, A_1(s), \qquad t - \tau < s \leqslant t, \tag{5}$$

with $Y(t, t) = I$ and $Y(s, t) = 0$ when $s > t$, and $x_0(t)$ is the solution of (1) with $x(t) = X(t)$, $-\tau \leqslant t \leqslant 0$, and $B(s)$ identically zero for all $s$.

Hence we have the response in $R^{n+1}$ given by

$$y(t) = y_0(t) + \int_0^t \bar{Y}(s, t)\, \Phi\{u(s)\}\, ds, \tag{6}$$

where

$$\bar{Y}(s,t) = \begin{bmatrix} Y(s,t)\,B(s) & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Phi\{u(s)\} = \begin{bmatrix} u(s) \\ \phi(u(s)) \end{bmatrix}, \qquad y_0(t) = \begin{bmatrix} x_0(t) \\ 0 \end{bmatrix}.$$

## Properties of the Optimal Control

Definition. The set of attainability $K(t_1)$ is the set of all end-points $y(t_1)$ in $R^{n+1}$ corresponding to controls $u(t) \in U$ on $0 \leqslant t \leqslant t_1$.

Theorem. *The set of attainability $K(t_1)$ is compact, convex, and varies continuously with $t_1$ on $t_1 \geqslant 0$.*

*Proof.* The convexity and compactness results can be proved by adopting the approach used in Neustadt [4]. The continuity property follows by using the procedure of Lee and Markus [7].

Theorem. *If there is a measurable control satisfying conditions* (2) *and* (3) *and with response $x(t)$ giving $x(t_1) = \eta$ for some $t_1 \geqslant 0$, then there exists an optimal control.*

*Proof.* Define the set $l$ as follows:

$$l = \{(\eta, \xi) \colon 0 \leqslant \xi \leqslant L\},$$

and is compact.

Following [1] we define $t^*$ as the greatest lower bound of all times $t_1$ such that $K(t_1)$ meets $l$. The continuous dependence of $K(t_1)$ on $t_1$ implies that the set of $t_1$ giving $K(t_1) \cap l$ nonempty is closed. Hence $t^*$ is the minimal time at which $K(t_1)$ meets $l$. Let $u^*(t) \in U$ on $0 \leqslant t \leqslant t^*$ be any control steering the system to $K(t^*) \cap l$. Then $u^*(t)$ is an optimal control. Q.E.D.

Define $K^* = K(t^*)$ and let $l^* = K^* \cap l$.

Lemma. *The set $l^*$ is contained in the boundary of $K^*$.*

*Proof.* Suppose the contrary, i.e., let $y^* \in l^*$, and suppose $y^*$ is an interior point of $K^*$. Then, since $K(t_1)$ varies continuously with $t_1$, there exists a $\hat{t} < t^*$ such that $y^*$ is an interior point of $K(\hat{t})$. But this contradicts the optimality of $u^*(t)$. Hence assertion is false. Q.E.D.

Theorem. *If the control $u^*(t)$ is a minimum time control with minimum time* *then* $t^*$, $t^* \geqslant 0$, *then there exists a constant* $p_0 \leqslant 0$ *and a nontrivial discontinuous (see Appendix) solution* $p(t)$ *of the system of equations,*

$$\begin{aligned}\dot{p}(t) &= -p(t)\,A_1(t) - p(t+\tau)\,A_2(t+\tau) + \dot{p}(t+\tau)\,A_3(t+\tau)\\ &\quad + p(t+\tau)\,\dot{A}_3(t+\tau), \qquad 0 \leqslant t \leqslant t^* - \tau,\\ \dot{p}(t) &= -p(t)\,A_1(t), \qquad\qquad t^* - \tau < t \leqslant t^*,\end{aligned}$$

*such that*

$$p(t)\,B(t)\,u^*(t) + p_0\phi\{u^*(t)\} = \max_{u\in U}\{p(t)\,B(t)\,u + p_0\phi(u)\},$$

*for almost all* $t$ *in* $[0, t^*]$.

*Proof.* Let $y^*$ be a point in $l^*$. Then, by the lemma, $y^*$ is a boundary point of $K^*$. Consequently, since $K^*$ is convex, there exists a supporting hyperplane at $y^*$ such that

$$\lambda y^* \geqslant \lambda y, \qquad \text{for all } y \in K^*, \tag{7}$$

where $\lambda$ defines the normal to the hyperplane, and has the property $\lambda_{n+1} \leqslant 0$ [4, Theorem 5].

It will now be shown that (7) implies the condition

$$\lambda \bar{Y}(t, t^*)\,\Phi\{u^*(t)\} = \max_{u\in U}\{\lambda \bar{Y}(t, t^*)\,\Phi(u)\} \qquad \text{for almost all } t \text{ in } [0, t^*]. \tag{8}$$

Assume (7) is false: that is, suppose that,

$$\lambda \bar{Y}(t, t^*)\,\Phi\{u^*(t)\} < \max_{u\in U}\{\lambda \bar{Y}(t, t^*)\,\Phi(u)\}$$

for some set of positive duration in $[0, t^*]$.

Now define the control $\bar{u}(t)$ on $[0, t^*]$ as follows:

$$\lambda \bar{Y}(t, t^*)\,\Phi\{\bar{u}(t)\} = \max_{u\in U}\{\lambda \bar{Y}(t, t^*)\,\Phi(u)\}$$

Hence

$$\lambda \bar{y}(t^*) > \lambda y^*.$$

This last result gives a contradiction and hence (8) is true.

Using (6) in (8) gives

$$\begin{aligned}&\hat{\lambda} Y(t, t^*)\,B(t)\,u^*(t) + \lambda_{n+1}\phi\{u^*(t)\}\\ &\qquad = \max_{u\in U}\{\hat{\lambda} Y(t, t^*)\,B(t)\,u + \lambda_{n+1}\phi(u)\}, \text{ for almost all } t \text{ in } [0, t^*],\end{aligned}$$

where

$$\hat{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n).$$

Now define a row vector $p(t)$ as follows:

$$p(t) = \hat{\lambda} Y(t, t^*).$$

Differentiating and using (4) and (5) gives the required results.
Identifying $\lambda_{n+1} = p_0$ completes the proof of the theorem.

## Appendix

The matrix $Y(s, t)$ given by (4)–(5) must satisfy the following conditions:

$$Y(\{t - \tau)+, t) - Y(\{t - \tau\}-, t) = -A_3(t),$$

$$Y(\{t - k\tau\}+, t) - Y\{t - k\tau\}-, t)$$
$$= [Y(\{t - \overline{k - 1}\,\tau\}+, t) - Y(\{t - \overline{k - 1}\,\tau)-, t)]\, A_3(t - \overline{k - 1}\,\tau),$$

where $k = 1, 2,..., K$ (see below).

The above discontinuities in $Y(s, t)$ lead to the following discontinuities for $p(t)$:

$$p(m_K+) - p(m_K-) = -p(t^*-)$$

$$p(m_k+) - p(m_k-) = \{ p(m_{k+1}+) - p(m_{k-1}-)\}\, A_3(m_{k+1}), \quad k = 1, 2,..., K,$$

where $m_k = t^* - \{K + 1 - k\}\tau$, $1 \leqslant k \leqslant K$, and where $K$ is specified by the condition $K\tau \leqslant t^* < (k + 1)\tau$.

## References


1. L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkredelidze, and E. F. Mishchenko, "The Mathematical Theory of Optimal Processes," Pergamon Press, London, 1946.
2. M. N. Oguztoreli, "Time-Lag Control Systems," Academic Press, New York, 1966.
3. H. T. Banks and M. Q. Jacobs, The optimization of trajectories of linear functional differential equations, *SIAM J. Control* 8 (1970), 461–488
4. L. W. Neustadt, Time optimal control systems with position and integral limits, *J. Math. Anal. Appl.* 3 (1961), 406–427.
5. L. W. Neustadt, The existence of optimal controls in the absence of convexity conditions, *J. Math. Anal. Appl.* 7 (1963), 110–117.
6. J. K. Aggarwal and D. H. Eller, Optimal control of neutral systems, *Internat. J. Control* 14 (1971), 309–319.
7. E. B. Lee and L. Markus, "Foundations of Optimal Control Theory," Wiley, New York, 1967.


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